

# BOUNDING $j$ -INVARIANT OF INTEGRAL POINTS ON CERTAIN MODULAR CURVES

MIN SHA

ABSTRACT. In this paper, we obtain two effective bounds for the  $j$ -invariant of integral points on certain modular curves which has positive genus and less than three cusps.

## 1. INTRODUCTION

Let  $\Gamma$  be a congruence subgroup of level  $N$  ( $N \geq 2$ ) and  $X_\Gamma$  its corresponding modular curve. Assume that  $X_\Gamma$  is defined over a number field  $K$ . Let  $S$  be a finite set of absolute values of  $K$ , containing all the Archimedean valuations and normalized with respect to  $\mathbb{Q}$ . We call a  $K$ -rational point  $P \in X_\Gamma(K)$  an  $S$ -integral point if  $j(P) \in \mathcal{O}_S$ , where  $j$  is the standard  $j$ -invariant function on  $X_\Gamma$  and  $\mathcal{O}_S$  is the ring of  $S$ -integers in  $K$ .

We use the standard notation  $\nu_\infty$  for the number of cusps of  $X_\Gamma$ . By classical Siegel's finiteness theorem [8],  $X_\Gamma$  has only finitely many  $S$ -integral points when  $X_\Gamma$  has positive genus or  $\nu_\infty \geq 3$ . But the existing proofs of Siegel's theorem are not effective, that is they don't provide with any effective bounds for the  $j$ -invariant of  $S$ -integral points.

The most recently Sha [6] gave some effective bounds for the  $j$ -invariant of integral points on arbitrary modular curves over arbitrary number fields assuming that  $\nu_\infty \geq 3$ . Especially, in the non-split Cartan case Bajolet and Sha [1] gave much better bounds. But there exist some exceptional cases, such as  $X_0(p)$  for a prime  $p > 7$  and  $p \neq 13$ , it has two cusps with positive genus, so it satisfies Siegel's theorem but not satisfies the condition in [6]. In this paper, we will deal with certain exceptional cases.

Let  $\mathcal{H}$  denote the Poincaré upper half-plane:  $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im}\tau > 0\}$ . Notice that the curve  $X_\Gamma$  has finitely many elliptic points. We assume that the set of its elliptic points is  $\{P_1, P_2, \dots, P_n\}$ . For each elliptic point  $P_i$ , we fix a pre-image  $z_i$  in  $\mathcal{H}$ . We denote by  $\Gamma_{z_i}$  the stabilizer of  $z_i$  in  $\Gamma$ . Let  $\tilde{\Gamma}$  be the congruence subgroup generated by  $\Gamma(N)$  and  $\{\Gamma_{z_1}, \dots, \Gamma_{z_n}\}$ . Since for a point  $P \in X_{\tilde{\Gamma}}$  fix a pre-image  $z \in \mathcal{H}$ , the index  $[\pm\Gamma_z : \pm\tilde{\Gamma}_z]$  doesn't depend on the choice of  $z$  and is equal to the ramification index of  $P$  over  $X_\Gamma$ , then the natural finite covering  $\phi : X_{\tilde{\Gamma}} \rightarrow X_\Gamma$  is unramified outside the cusps.

Notice that if there exists a finite covering  $X_{\Gamma'} \rightarrow X_\Gamma$  which is unramified outside the cusps such that  $X_{\Gamma'}$  has at least three cusps, then the finite covering  $X_{\tilde{\Gamma}} \rightarrow X_\Gamma$  also satisfies that  $X_{\tilde{\Gamma}}$  has at least three cusps. Under this assumption, by the

---

2010 *Mathematics Subject Classification*. Primary 11G16, 11J86; Secondary 14G35, 11G50.

*Key words and phrases*. Modular curves,  $j$ -invariant, integral points.

The author is supported by China Scholarship Council.

results in [6] we can get an effective Siegel's theorem for  $X_{\tilde{\Gamma}}$ . Then the effective Siegel's theorem for  $X_{\Gamma}$  follows from quantitative Riemann existence theorem [3] and quantitative Chevalley-Weil theorem [4].

Before stating the main results, we would like to give two examples satisfying our assumptions.

**Example 1.1.**  $X_0(p)$  for a prime  $p > 7$  and  $p \neq 13$ , it has two cusps with positive genus. By [2, Proof of Theorem 10], it has a finite covering which is étale outside the cusps and has at least three cusps.

**Example 1.2.** Assume that  $\Gamma_{z_1}, \dots, \Gamma_{z_n}$  generate a finite subgroup  $G$  and  $|G| < \frac{1}{4}N^2 \prod_{q|N} (1 - q^{-2})$ , where the product being taken over all primes  $q$  dividing  $N$ . By [7, Corollary 2.4],  $X_{\tilde{\Gamma}}$  has at least three cusps.

First we fix some notations. Put

$$d_N = \begin{cases} \frac{1}{2}N^3 \prod_{q|N} (1 - 1/q^2) & \text{if } N > 2, \\ 6 & \text{if } N = 2, \end{cases}$$

where  $q$  runs through all primes dividing  $N$ . Let  $d = [K : \mathbb{Q}]$  and

$$D^* = D^{d_N} e^{(h(S) + (1 + \log 1728)\Lambda)dd_N},$$

where  $D$  is the absolute discriminant of  $K$ ,

$$\Lambda = \left( \left( \frac{d_N(N-6)}{12N} + 2 \right) d_N \right)^{25 \left( \frac{d_N(N-6)}{12N} + 2 \right) d_N},$$

and

$$h(S) = \frac{\sum_{v \in S} \log \mathcal{N}_{K/\mathbb{Q}}(v)}{d},$$

the norm  $\mathcal{N}_{K/\mathbb{Q}}(v)$  of a place  $v$  is the absolute norm of the corresponding prime ideal if  $v$  is finite, and is set to be 1 when  $v$  is infinite. Now we define

(1.1)

$$\Delta = (dd_N)^{-dd_N} \sqrt{N^{Ndd_N} |D^*|^N} (\log(N^{Ndd_N} |D^*|^N))^{Ndd_N} \left( \prod_{\substack{v \in S \\ v \nmid \infty}} \log \mathcal{N}_{K/\mathbb{Q}}(v) \right)^{Nd_N}.$$

In addition, we denote by  $p$  the maximal rational prime below  $S$ , with the convention  $p = 1$  if  $S$  consists only of the infinite places.

We denote by  $h(\cdot)$  the usual absolute logarithmic height. For  $P \in X_{\Gamma}(\bar{K})$ , we write  $h(P) = h(j(P))$ . Now we are ready to state the main results.

**Theorem 1.3.** Assume that  $X_{\Gamma}$  has a finite covering which is unramified outside the cusps and has at least three cusps, and  $N$  is not a power of any prime. Then for any  $S$ -integral point  $P$  on  $X_{\Gamma}$ , we have

$$h(P) \leq (Cds_N^2 N^2)^{2sNd_N} (\log(dNd_N))^3 p^{dNd_N} \Delta,$$

where  $C$  is an absolute effective constant.

When  $N$  is a prime power, we define

$$M = \begin{cases} 2N & \text{if } N \text{ is not a power of } 2, \\ 3N & \text{if } N \text{ is a power of } 2. \end{cases}$$

**Theorem 1.4.** *Assume that  $X_\Gamma$  has a finite covering which is unramified outside the cusps and has at least three cusps, and  $N$  is a power of some prime. Then for any  $S$ -integral point  $P$  on  $X_\Gamma$ , we can get an upper bound for  $h(P)$  by replacing  $N$  by  $M$  in Theorem 1.3.*

## 2. QUANTITATIVE RIEMANN EXISTENCE THEOREM FOR $X_{\tilde{\Gamma}}$

The Riemann Existence Theorem asserts that every compact Riemann surface is (analytically isomorphic to) a complex algebraic curve. Bilu and Strambi [3, Theorem 1.2] gave a quantitative version of Riemann Existence Theorem, which is a key tool in this paper.

Notice that the  $j$ -invariant induces naturally two coverings  $X_\Gamma \rightarrow \mathbb{P}^1(\mathbb{C})$  and  $X_{\tilde{\Gamma}} \rightarrow \mathbb{P}^1(\mathbb{C})$ , we use the same notation  $j$  to denote both of them without confusions. In addition, the  $j$ -invariant also defines an isomorphism  $X(1) \cong \mathbb{P}^1(\mathbb{C})$ .

For the covering  $j : X_{\tilde{\Gamma}} \rightarrow \mathbb{P}^1(\mathbb{C})$ , we assume that its degree is  $\tilde{n}$  and the genus of the curve  $X_{\tilde{\Gamma}}$  is  $\tilde{g}$ . Then there exists a rational function  $y \in \bar{K}(X_{\tilde{\Gamma}})$  such that  $\bar{K}(X_{\tilde{\Gamma}}) = \bar{K}(j, y)$  and the rational functions  $j, y \in \bar{K}(X_{\tilde{\Gamma}})$  satisfy the equation  $\tilde{f}(j, y) = 0$ , where  $\tilde{f}(j, y) \in \bar{K}[X, Y]$  is an absolutely irreducible polynomial satisfying

$$(2.1) \quad \deg_X \tilde{f} = \tilde{g} + 1, \quad \deg_Y \tilde{f} = \tilde{n}.$$

For the natural sequence of coverings  $X(N) \rightarrow X_{\tilde{\Gamma}} \rightarrow \mathbb{P}^1(\mathbb{C})$ , applying the formula in the bottom of [5, Page 101] and the genus formula of  $X(N)$  (see [5, Figure 3.4]) respectively, we have

$$(2.2) \quad \tilde{n} \leq d_N, \quad \tilde{g} \leq 1 + \frac{d_N(N-6)}{12N}.$$

## 3. QUANTITATIVE CHEVALLEY-WEIL THEOREM FOR $\phi : X_{\tilde{\Gamma}} \rightarrow X_\Gamma$

The Chevalley-Weil theorem asserts that for an étale covering of projective varieties over a number field  $F$ , the discriminant of the field of definition of the fiber over an  $F$ -rational point is uniformly bounded. Bilu, Strambi and Surroca [4] got a fully explicit version of this theorem in dimension 1, which is another key tool of this paper.

For the covering  $j : X_\Gamma \rightarrow \mathbb{P}^1(\mathbb{C})$ , since there are only two elliptic points  $\mathrm{SL}_2(\mathbb{Z})i$  and  $\mathrm{SL}_2(\mathbb{Z})e^{2\pi i/3}$  of  $X(1)$ , it is unramified outside the two points  $j(i) = 1728$  and  $j(e^{2\pi i/3}) = 0$ . For the covering  $X_{\tilde{\Gamma}} \rightarrow X_\Gamma$ , it is unramified outside the cusps. Notice that the poles of  $j$ -invariant are exactly the cusps. Then by [4, Theorem 1.6], for every  $P \in X_\Gamma(\bar{K})$  and  $\tilde{P} \in X_{\tilde{\Gamma}}(\bar{K})$  such that  $\phi(\tilde{P}) = P$ , we have

$$(3.1) \quad \mathcal{N}_{K/\mathbb{Q}}(D_{K(\tilde{P})/K}) \leq e^{[K(\tilde{P}):\mathbb{Q}] \cdot (h(S) + (1 + \log 1728)\tilde{\Lambda})},$$

where  $\tilde{\Lambda} = ((\tilde{g} + 1)\tilde{n})^{25(\tilde{g}+1)\tilde{n}}$ . According to (2.2), we have  $\tilde{\Lambda} \leq \Lambda$ . Hence

$$(3.2) \quad \mathcal{N}_{K/\mathbb{Q}}(D_{K(\tilde{P})/K}) \leq e^{[K(\tilde{P}):\mathbb{Q}] \cdot (h(S) + (1 + \log 1728)\Lambda)}.$$

Notice that the degree  $[K(\tilde{P}) : K]$  is not greater than the degree of  $\phi$ , so we have  $[K(\tilde{P}) : K] \leq d_N$ .

#### 4. PROOF OF THEOREMS

In this section we suppose that the curve  $X_{\tilde{F}}$  has at least three cusps.

Let  $K_0 = K(\tilde{P})$  and  $d_0 = [K_0 : \mathbb{Q}]$ . Let  $S_0$  be the set consisting of the extensions of the places from  $S$  to  $K_0$ , i.e.

$$S_0 = \{v \in M_{K_0} : v|v_0 \in S\},$$

where  $M_{K_0}$  is the set of all valuations (or places) of  $K_0$  extending the standard infinite and  $p$ -adic valuations of  $\mathbb{Q}$ . Put  $s_0 = |S_0|$ . We define the following quantity

$$(4.1) \quad \Delta_0 = d_0^{-d_0} \sqrt{N^{d_0 N} |D_0|^N} (\log(N^{d_0 N} |D_0|^N))^{d_0 N} \left( \prod_{\substack{v \in S_0 \\ v \nmid \infty}} \log \mathcal{N}_{K_0/\mathbb{Q}}(v) \right)^N,$$

where  $D_0$  is the absolute discriminant of  $K_0$ .

Notice that  $d_0 \leq dd_N$  and  $s_0 \leq sd_N$ . Let  $D_{K_0/K}$  be the relative discriminant of  $K_0/K$ . By Formula (3.2), we have

$$\begin{aligned} D_0 &= \mathcal{N}_{K/\mathbb{Q}}(D_{K_0/K}) D^{[K_0:K]} \\ &\leq D^*. \end{aligned}$$

Now let  $v_0$  be a non-archimedean place of  $K$ , and  $v_1, \dots, v_m$  all its extensions to  $K_0$ , their residue degrees over  $K$  being  $f_1, \dots, f_m$  respectively. Then  $f_1 + \dots + f_m \leq [K_0 : K] \leq d_N$ , which implies that  $f_1 \cdots f_m \leq 2^{d_N}$ . Since  $\mathcal{N}_{K_0/\mathbb{Q}}(v_k) = \mathcal{N}_{K/\mathbb{Q}}(v_0)^{f_k}$  for  $1 \leq k \leq m$ , we have

$$\prod_{v|v_0} \log \mathcal{N}_{K_0/\mathbb{Q}}(v) \leq 2^{d_N} (\log \mathcal{N}_{K/\mathbb{Q}}(v_0))^{d_N}.$$

Hence

$$(4.2) \quad \prod_{\substack{v \in S_0 \\ v \nmid \infty}} \log \mathcal{N}_{K_0/\mathbb{Q}}(v) \leq 2^{sd_N} \left( \prod_{\substack{v \in S \\ v \nmid \infty}} \log \mathcal{N}_{K/\mathbb{Q}}(v) \right)^{d_N}.$$

So we have

$$\Delta_0 \leq 2^{sNd_N} \Delta.$$

First we assume that  $N$  is not a power of any prime. By [6, Theorem 1.2] we have

$$h(\tilde{P}) \leq (Cd_0 s_0 N^2)^{2s_0 N} (\log(d_0 N))^3 p^{d_0 N} \Delta_0,$$

where  $C$  is an absolute effective constant. Note that  $h(P) = h(\tilde{P})$ , then we have

$$(4.3) \quad h(P) \leq (Cdsd_N^2 N^2)^{2sNd_N} (\log(dNd_N))^3 p^{dNd_N} \Delta,$$

the constant  $C$  being modified. So we prove Theorem 1.3.

For the case that  $N$  is a prime power, applying [6, Theorem 1.2], we can easily prove Theorem 1.4.

# ACKNOWLEDGEMENT

The author would like to thank his advisor Yuri Bilu for lots of stimulating suggestions, helpful discussions and careful reading.

# REFERENCES

- [1] A. Bajolet and M. Sha, *Bounding  $j$ -invariant of integral points on  $X_{\text{ns}}^+(p)$* , submitted; arXiv:1203.1187v2.
- [2] Yu. Bilu, *Baker's method and modular curves*, A Panorama of Number Theory or The View from Baker's Garden (edited by G. Wüstholz), 73-88, Cambridge University Press, 2002.
- [3] Yu. Bilu and M. Strambi, *Quantitative Riemann existence theorem over a number field*, Acta Arith. **145** (2010), 319-339.
- [4] Yu. Bilu, M. Strambi and A. Surroca, *Quantitative Chevalley-Weil Theorem for Curves*, submitted; arXiv:0908.1233v3.
- [5] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Springer, New York, 2005.
- [6] M. Sha, *Bounding  $j$ -invariant of integral points on modular curves*, submitted; arXiv:1208.1337v2.
- [7] Yu. Bilu and M. Illengo, *Effective Siegel's Theorem for Modular Curves*, Bull. London Math. Soc. **43** (2011), 673-688.
- [8] C.L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Pr. Akad. Wiss. (1929), no. 1. (=Ges. Abh. I, 209-266, Springer, 1966.)

INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ BORDEAUX 1 , 33405 TALENCE  
CEDEX, FRANCE

*E-mail address:* shamini2010@gmail.com